

Introduction to Orbital Mechanics

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1 Introduction

In introductory physics classes, we learn Newton's laws of motion and his law of gravity, as well as Kepler's laws for planetary motions. However, we usually make some simplifications which do not necessarily reflect on the true picture. Prominent among these simplifications are:

1. All bodies are represented as point masses.
2. The gravitational field of a body is spherically symmetric.
3. The central body (Sun or Earth) is fixed in space.
4. There are only two bodies in any given problem, one of which doesn't move.
5. Masses don't change.

The object of this class is to investigate these restrictions in the setting of satellites going around earth, however, we could easily extend this to the solar system.

One other limitation of the way we study orbital motion in elementary physics is, that we look at the planet circling the sun from an outside vantage point as if it were a clock work model. However, in reality we are sitting on one of the bodies and are a moving observer. The connections between the simple motions of earth and its satellite and the observations will also be studied.

2 Newton's Laws of motion and Gravity — Kepler's Laws, simple examples

2.1 Newton's Laws

All of you have previously encountered Newton's laws of motion. These laws form the basis of the part of mechanics which is known as dynamics. They are as follows:

1. A body does not change its state of rest or uniform straight-line motion unless it is compelled by some force to change that state.
2. The change of motion is proportional to the force and takes place in the same direction as the force.
3. Action is always contrary and equal to reaction.

The first and second laws of motion are usually combined in a single vector equation:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}, \quad (1)$$

where \mathbf{F} is the force, the scalar m is the mass of the body, and \mathbf{a} its acceleration. This simplified version assumes tacitly that the mass is a constant. Often, in our daily life experiences, this is a valid assumption or at least a good approximation. But even, when you press the accelerator in a car the mass of the car changes due to the burning of fuel and rubber. However, this change is insignificant compared with the mass of the vehicle and can be neglected.

Accelerating rockets is always accompanied by huge changes of the mass of the rocket itself. Furthermore, since space travel is often done at very high velocities, relativity effects can play a role, albeit a very minor one in the context of this class. To account for this we use the following form of Newton's law:

$$\mathbf{F} = \frac{dm\mathbf{v}}{dt}, \quad (2)$$

where \mathbf{v} is the velocity of the body. It is easy to see that (1.2) implies (1.1) for a constant mass. Newton also published his law of gravity

$$\mathbf{F} = \frac{GMm}{|\mathbf{r}|^3}\mathbf{r}, \quad (3)$$

where \mathbf{F} is the attracting force between the two bodies of mass M and mass m and \mathbf{r} is a vector pointing from one body to the other. The constant G is the Gravitational constant which is measured at

$$G = 6.672 \times 10^{-11} \text{m}^3/\text{kg}\cdot\text{s}^2.$$

If the bigger of the two bodies were fixed at the origin of a coordinate system, and the mass m of the smaller body were fixed, we could describe the motion of the smaller body by

$$m \frac{d\mathbf{v}}{dt} = -\frac{GMm}{|\mathbf{r}|^3} \mathbf{r} \quad (4)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad (5)$$

a coupled system of two first order ordinary differential equations. Since \mathbf{r} and \mathbf{v} are vectors, it is actually a coupled system of six ordinary differential equations. Given initial conditions \mathbf{r}_0 and \mathbf{v}_0 we can describe all future positions of the body of mass m . This tells us that we usually need seven variables to describe the motion completely, time, three spatial components and three velocity components.

How does the above model comply with Newton's third law? The above model is really a model for the motion of a single body in a field which has spherical symmetry. The second body of mass M is completely ignored. However, if we have two bodies, which are connected by a symmetric force (which Newton's third law implies) any motion of the smaller body will be accompanied by a motion of the larger body. We will, however, prove that in many cases this motion of the larger body can be ignored. But we will also give several significant examples, where it can't be.

2.2 Kepler's laws

Newton discovered his law of gravity around 1665. This gave finally an explanation of the motion of planets around the sun. Before Newton several quantitative descriptive models were given. First there was the Greek geocentric model using epicycles, a very complicated model, but highly successful in predicting the positions of planets for over 2000 years. Copernicus gave a heliocentric model, still insisting that planets move on circles. The most concise of these descriptive models was given around 1600 by Johannes Kepler. Precise observations by Tycho Brahe in the second half of the 16th century dispelled myth of circular motion. Based on these observations Kepler spelled out the following laws:

1. Planets move around the sun in elliptic orbits with the sun located at one focus of the ellipse.
2. As the planet moves in its orbit around the sun, equal areas as measured from the focus are swept out in equal times.

3. The square of the period of the orbit is proportional to the cube of the semimajor axis of the ellipse.

We will prove these laws from Newton's laws later in the course. Kepler's laws are purely descriptive and make no attempt to explain the motion. For this reason they are a more practical tool to study the motions of bodies in a gravitational field. Kepler's laws still allow for circular motions, however, they are not restricted to these motions. Nevertheless, most of the planets, the moon and many artificial satellites actually travel along nearly circular orbits.

2.3 Circular Orbits

In this section we assume that the central body of mass M_E is the earth and is fixed (or moving uniformly). And the second body of mass m is a small artificial satellite. The Earth's mass $M_E = 5.9742 \times 10^{24}$ kg. Sputnik I on the other hand weighed 83.6 kg. So its gravitational pull on the Earth can probably be neglected. (This is certainly not true for its effect on world history or American politics). We will also assume that the satellite moves on a circular orbit. And that earth is at the center of this orbit. Let r be the radius of the circular orbit. The distance traveled after a time t is given by

$$s(t) = r(\phi(t) - \phi(0)).$$

Hence,

$$v = r\phi' = r\omega$$

The area swept in a time t is given by

$$A(t) = r^2\phi(t).$$

Kepler's second law implies that this area is proportional to t . It follows that

$$\phi(t) = \omega t$$

for some constant angular velocity ω . Thus we have the rule:

Satellites on circular orbits must move at a constant angular velocity.

In polar coordinates we have then

$$\mathbf{r}(t) = \begin{pmatrix} r \cos \omega t \\ r \sin \omega t \end{pmatrix},$$

and therefore

$$\mathbf{r}''(t) = -\omega^2 \begin{pmatrix} r \cos \omega t \\ r \sin \omega t \end{pmatrix} = -\omega^2 \mathbf{r}(t).$$

(4) can thus be written as

$$m\omega^2\mathbf{r}(t) = \frac{GM_E m}{r^3}\mathbf{r}(t),$$

or simply as

$$\omega^2 r^3 = GM_E. \tag{6}$$

If T is the period of this motion then

$$\omega T = 2\pi.$$

substituting this into (6) we get

$$\frac{r^3}{T^2} = \frac{GM_E}{4\pi^2}, \tag{7}$$

this is of course Kepler's third law in the special case of circular orbits. The expression GM_E is called the geocentric gravitational constant. Its value is

$$GM_E = 3986004 \times 10^8 \text{m}^3/\text{s}^2.$$

Example: A satellite on circular orbit, has a period of 90 min. What is radius of the orbit and the altitude of the satellite. From (7) we get:

$$r = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}} = 6652555.5\text{m} = 6652.6\text{km}.$$

The radius of the Earth is about 6360 km, hence the altitude is about 292 km.

2.4 Homework:

Communications satellites should stay in or close to the same place of the sky all the time. Otherwise one would have to constantly move the satellite dish to get good reception. Such orbits are called geo-stationary. Assume a circular geo-stationary orbit in the equatorial plane of the earth.

1. Find the period and the radius of this orbit.
2. Assume that the satellite is at a fixed longitude of 118° west (same as LA) in what direction do you have to point your dish if you live in LA (32° north).
3. For the same satellite what direction do you have to point your dish if you live in Las Vegas (33° north and 112° west.)?

3 The two body problem

3.1 Circular Orbits

The object of this section is to correct some of the aspects of the computations in the preceding section. We first drop the assumption that the central body of mass M is at rest or moves uniformly. We replace with the following more sensible assumption.

Assumption: *The center of mass of the two body system consisting is at rest or moves uniformly.*

One can always arrive at this assumption by decomposing the motion of the two body system into two pieces, a motion of the center of mass and a motion of the two components with respect to the center of mass. If we assume that there is no external forces (except the gravitational forces between the two bodies), Newton's first law tells us that the center of mass is either at rest or moves uniformly. In the case of a satellite moving around Earth, this assumption is not quite correct, since the center of mass moves in the gravitational field of the sun, hence it is not a uniform motion with constant speed. However, on the surface of the Earth the acceleration due to the gravitational pull of the sun is approximately $.0059m/s^2$ compared to Earth gravity of $9.81m/s^2$, and we will ignore it for the time being.

We center our coordinate system at the center of gravity, which sits on the line connecting the two bodies. In circular motion the distance between the two bodies is constant r . We have $r = r_1 + r_2$, where r_1 is the distance of the body of mass M from the center of mass, and r_2 is the distance of the body of mass m from the center of mass. We have from the definition of the center of mass:

$$Mr_1 = mr_2.$$

Furthermore, we have the following balance of centrifugal forces:

$$\begin{aligned}mr_2\omega^2 &= G\frac{Mm}{r^2} \\ Mr_1\omega^2 &= G\frac{Mm}{r^2}\end{aligned}$$

Adding these equations and using the property of the center of mass we get:

$$r^3\omega^2 = (r_1 + r_2)^3\omega^2 = G(M + m),$$

or

$$r^3\omega^2 = GM\left(1 + \frac{m}{M}\right) \tag{8}$$

This is the corrected form of Kepler's third law for circular orbits. One sees that this is now dependent on the mass of the satellite. In the case Sputnik I, the correction factor

$$\frac{m}{M} = \frac{83.6}{5.9742 \times 10^{24}} = 1.44 \times 10^{23}$$

which can be definitely ignored.

Examples: The ratio of the mass of the moon to the mass of the earth is

$$\frac{M_L}{M_E} = 0.0123.$$

The radius of the moons nearly circular orbit is about 384400 km. This implies that $r_1 = 0.0123 \times 384400 = 472$ i.e the center of the rotation is 472 km above the center of the earth. The orbital period of the moon is

$$27.3 \times 86400 = 2358720s$$

i. e. $\omega = 2.664 \times 10^{-6} s^{-1}$. This produces an acceleration pointed away from the center of the earth and opposite the center of the earth of

$$\omega^2(R_E + r_1) = 4.86 \times 10^{-5} m/s^2,$$

which is responsible for the second tide.

Another more exciting example is from the search for extra solar planets. Planets do not radiate any energy, and it is therefore impossible to directly observe planets circling around stars. However, if a large planet of mass m_P circles a star of mass M_S with period T we have the following:

$$\begin{aligned} r &= r_S + r_P \\ M_S r_S &= m_P r_P \\ \frac{r^3}{T^2} &= \frac{G}{4\pi^2} (M_S + m_P) \end{aligned}$$

The star will radiate on a different frequency, depending on its velocity relative to the observer. If the star moves on a circular path (in the same plane with the observer) with speed $v = r_S \omega = 2\pi r_S / T$, the maximal difference between velocities is $2v$. This can be computed from the shift in frequencies. One can also directly observe the period T . From this one can compute the ratio of the two masses and r_S . Often one can also estimate M_S from the total radiation of the star. This then completely determines the orbit and size of the planet.

3.2 The General two Body Problem I — Elementary Conservation Laws

In this section we will investigate the two body problem in more general terms. To do this consider the situation in the figure below. There two bodies of masses m_1 and m_2 are at

locations \mathbf{r}_1 and \mathbf{r}_2 . We also assume that the only acting force is the gravitational attraction between the two bodies:

$$\mathbf{F}_{21} = G \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2) = G \frac{m_1 m_2}{r^3} (\mathbf{r}_1 - \mathbf{r}_2),$$

is the force acting on the body m_2 , and $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is the force acting on m_1 .

Newton's law implies

$$m_1 \frac{d^2 \mathbf{r}_1}{dt^2} = \mathbf{F}_{12} \tag{9}$$

$$m_2 \frac{d^2 \mathbf{r}_2}{dt^2} = \mathbf{F}_{21} \tag{10}$$

Adding these two equations we get:

$$0 = m_1 \frac{d^2 \mathbf{r}_1}{dt^2} + m_2 \frac{d^2 \mathbf{r}_2}{dt^2} = \frac{d}{dt} \left(m_1 \frac{d\mathbf{r}_1}{dt} + m_2 \frac{d\mathbf{r}_2}{dt} \right),$$

or

$$\mathbf{P} = m_1 \frac{d\mathbf{r}_1}{dt} + m_2 \frac{d\mathbf{r}_2}{dt}$$

is constant. This is known as conservation of momentum. It can be further interpreted as $\frac{d\rho}{dt} = 0$, where

$$\rho = \frac{m_1}{m_1 + m_2} \mathbf{r}_1 + \frac{m_2}{m_1 + m_2} \mathbf{r}_2,$$

is the location of the center of mass of the two bodies. We have does established

Conservation of Linear Momentum:

The center of mass of a two body system with no exterior forces moves at a uniform speed.

Since the motion of the center of mass is trivial we will ignore it for the remainder of this derivation. We will put the the origin of our reference frame at the center of mass. (See the Figure below.)

To continue we multiply the equations motion by m_2 and m_1 respectively and subtract the from each other to get

$$\frac{d^2}{dt^2} (\mathbf{r}_1 - \mathbf{r}_2) = -\frac{G(m_1 + m_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2).$$

We can further simplify this by introducing r_1 and r_2 , where r_j is the distance between m_j and the center of mass and set

$$\mathbf{r}_1 - \mathbf{r}_2 = (r_1 + r_2) \hat{\mathbf{r}}$$

Furthermore, we introduce the *reduced masses*:

$$M_R(m_1) = \frac{m_2^2}{m_1 + m_2} \quad (11)$$

$$M_R(m_2) = \frac{m_1^2}{m_1 + m_2} \quad (12)$$

If we only consider the motion of the body m_1 , we can write:

$$m_1 \frac{d^2 \mathbf{r}_1}{dt^2} = -\frac{Gm_1 M_R(m_1)}{r_1^2} \hat{\mathbf{r}}$$

and if we consider the motion of m_2 :

$$m_2 \frac{d^2 \mathbf{r}_2}{dt^2} = -\frac{Gm_2 M_R(m_2)}{r_2^2} \hat{\mathbf{r}}$$

If one considers the motion of a small satellite of mass m around earth,. The reduced mass M_R is nearly equal to the mass of the earth and the problem reduces to a one body problem.

To continue we consider the motion of the mass m_1 :

$$m_1 \frac{d^2 \mathbf{r}_1}{dt^2} = -\frac{Gm_1 M_R}{r_1^3} \mathbf{r}_1$$

This equation can be skalar multiplied by $\frac{d\mathbf{r}_1}{dt}$ to get

$$\begin{aligned} \frac{m_1}{2} \frac{d}{dt} \left(\frac{d\mathbf{r}_1}{dt} \cdot \frac{d\mathbf{r}_1}{dt} \right) &= -\frac{Gm_1 M_R}{r_1^3} \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_1 \\ &= Gm_1 M_R \frac{d}{dt} \left(\frac{1}{r_1} \right). \end{aligned}$$

or

$$\frac{1}{2} m_1 v_1^2 - \frac{Gm_1 M_R}{r_1} = K. \quad (13)$$

The quantity on the left of this equation is called the *total energy*, and we have established:

Conservation of Total Energy:

The total energy of the two body problem is constant.