The Vis-Viva Equation

The potential energy for a satellite of mass *m* in a planetary gravitational field is

$$E_{potential} = -\frac{GMm}{r} = -\frac{\mu m}{r} \tag{1}$$

where *M* is the mass of the planet and $\mu = GM$. The kinetic energy is

$$E_{kinetic} = \frac{1}{2} m V^2 \tag{2}$$

where *V* is the speed of the satellite, i.e., $V^2 = \vec{r}\,' \cdot \vec{r}\,'$ or $V = |\vec{r}\,'|$ (3)

By the ***law of conservation of total energy***, the sum of the kinetic and potential energies does not change as the satellite moves about its orbit:

$$E_{potential} + E_{kinetic} = \frac{1}{2} m V^2 - \frac{\mu m}{r} = Constant \tag{4}$$

What this means is that if r_1 and V_1 are the position and speed of the satellite at one point of the orbit, and if r_2 and V_2 are the position and speed at a second point of the orbit, then

$$\frac{1}{2} m V_1^2 - \frac{\mu m}{r_1} = \frac{1}{2} m V_2^2 - \frac{\mu m}{r_2} \tag{5}$$

Canceling out the common factor of *m*,

$$\frac{1}{2} V_1^2 - \frac{\mu}{r_1} = \frac{1}{2} V_2^2 - \frac{\mu}{r_2} \tag{6}$$

This equation is true for any two points we choose along the orbit. If we let the position and speed at point 1 be *r* and *V* (instead of r_1 and V_1), and if we choose point 2 to be perigee, then equation (6) becomes

$$\frac{1}{2} V^2 - \frac{\mu}{r} = \frac{1}{2} V_{perigee}^2 - \frac{\mu}{r_{perigee}} \tag{7}$$

But we know already that

$$r_{perigee} = a(1-e) \text{ and } V_{perigee} = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}} \tag{8), (9}$$

Substituting equations (8) and (9) into equation (7),

$$\frac{1}{2} V^2 - \frac{\mu}{r} = \frac{1}{2} \frac{\mu}{a} \frac{1+e}{1-e} - \frac{\mu}{a(1-e)} \tag{10}$$

Factoring out the common $\mu / a(1-e)$ on the right hand side of (10)

$$\frac{1}{2} V^2 - \frac{\mu}{r} = \frac{\mu}{a(1-e)}\left[\frac{1}{2}(1+e) - 1\right] = \frac{\mu}{a(1-e)}\left[\frac{1}{2} + \frac{e}{2} - \frac{2}{2}\right] = -\frac{\mu}{2a} \tag{11}$$

Solving for V^2 we find that

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{12}$$

Equation (12) is called the ***Vis-Viva Equation***.