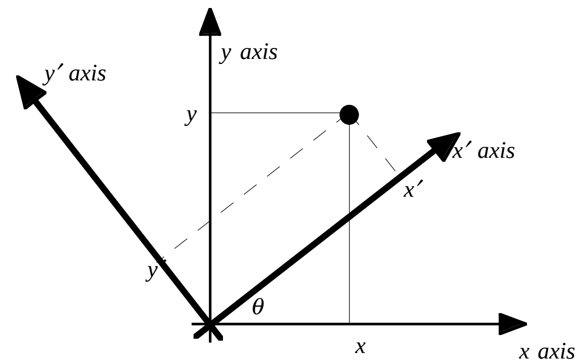
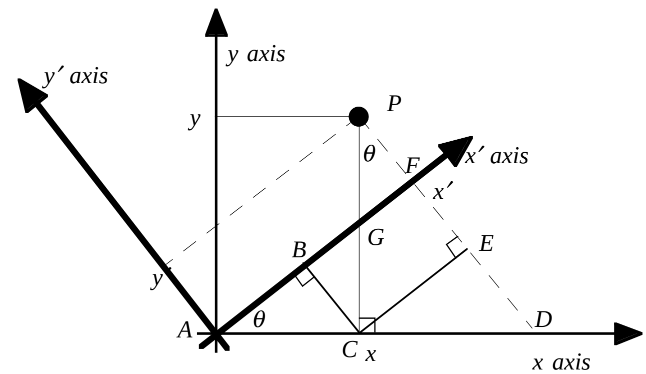
Rotation Matrices

Consider a rotation of the coordinate axes of the x-y plane about the origin by an angle θ. We are interested in finding the coordinates (x', y') of a point as measured in the new coordinate frame.



We determine the transformation by the following constructions:



The new x-coordinate is

$$x' = AB + BF \tag{1}$$

The old x-coordinate

$$x = AC \tag{2}$$

By definition of the cosine,

$$\cos\theta = \frac{AB}{AC} = \frac{AB}{x} \tag{3}$$

so that

$$AB = x\cos\theta \tag{4}$$

Thus (use 4 in 1)

$$x' = x\cos\theta + BF \tag{5}$$

Consider the right triangle *ACG*. Then because all of the angles in a triangle sum to π,

$$angle\ \ AGC = \pi / 2 - \theta \tag{6}$$

Thus in the right triangle *BCG*

$$angle\ \ GCB = \pi / 2 - angle(AGC) = \pi / 2 - (\pi / 2 - \theta) = \theta \tag{7}$$

Since *BC* is parallel to *PD*

$$angle(EPC) = angle(GCB) = \theta \tag{8}$$

From triangle EPC,

$$\sin\theta = \frac{CE}{PC} = \frac{CE}{y} = \frac{BF}{y} \quad (9)$$

Thus

$$BF = y\sin\theta \quad (10)$$

Substituting (10) into (5),

$$x' = x\cos\theta + y\sin\theta \quad (11)$$

Exercise: Show that

$$y' = y\cos\theta - x\sin\theta \quad (12)$$

Combining equations (11) and (12) we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is called a rotation matrix.

Exercise: Using a geometric argument, what is the rotation matrix that gives (x, y) in terms of (x', y'), i.e., try to find S so that $\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} x' \\ y' \end{pmatrix}$?

Exercise: What is $R(\theta)R(-\theta)$

Exercise: What is R^{-1} (the inverse of $R(\theta)$)?

Now consider a point $P = (x, y, z)$ in 3-dimensional space. Suppose we rotate the x-y plane around the z-axis by an angle q and call the new coordinates (x', y', z'). Since we are rotating about the z-axis, the value of z does not change, and the values of x' and y' are exactly the same as the ones we calculated above! Hence

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

$$z' = z$$

We can also write this in matrix form as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_z(\theta)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the rotation matrix for an angle θ about the *z* axis.

Exercise. Convince yourself that the rotation matrices for rotations about the x and y axes are given correctly by

$$R_x(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}$$

$$R_y(\theta)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$$

Now consider the effects of compound rotations: suppose that we first rotate by α about the *z* axis,

$$\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=R_z(\alpha)\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

we then rotate by β about the x' axis, giving us a third set of coordinates (x'',y'',z'').

$$\begin{pmatrix}x''\\ y''\\ z''\end{pmatrix}=R_{x'}(\beta)\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}$$

What is the relationship between the first and third set of coordinates?

$$\begin{pmatrix}x''\\ y''\\ z''\end{pmatrix}=R_{x'}(\beta)\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=R_{x'}(\beta)R_z(\alpha)\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

In general, the rotation matrix for a any sequence of rotations is just the product of the rotations in the reverse order. Fortunately, due to a theorem by Euler, we rarely have to multiply more than three rotation matrices together.

Euler's Theorem.

(1) The most general rotation of a rigid body (vector) with one point fixed is a rotation about some axis.

(2) Any sequence of rotations about the coordinate axes can be described by a single rotation matrix which is the product of the individual rotation matrices.

(3) Any rotation can be described by a product of at most three rotations about successive coordinate axes. The three angles are called ***Euler Angles***.

Remark. There sequence of rotations is not unique, i.e., you can get the same rotation in several different ways:

(a) A 3-1-3 rotation: rotate by α about *z*, β about x' and γ about z''

(b) A 1-2-3 rotation: rotate by α' about x, β' about y', and γ' about z''

etc.