Math 462 Homework Exercises (corrected)

1. Let $M = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$. Find: (a) det$M$; (b) The eigenvalues of $M$; (c) The eigenvectors of $M$; (d) The inverse of $M$

2. (Text 1.1) Show that $\dfrac{-1+i\sqrt{3}}{2}$ is a cube root of 1.

3. Find all $z \in \mathbb{C}$ such that $z^4 = 1$.

4. Prove or disprove the following statement: $\mathbb{C}$ is a vector space over $\mathbb{R}$.

5. Prove that

$$z^j - \lambda^j = (z-\lambda)(z^{j-1} + z^{j-2}\lambda + \cdots + z\lambda^{j-2} + \lambda^{j-1})$$

for all $j = 1, 2, ...$

6. (Text 4.3) Let $p(z), q(z) \in \mathbb{P}(\mathbb{F})$ with $p(z) \neq 0$ (the zero polynomial). We showed in class that there exist polynomials $s(z), r(z) \in \mathbb{P}(\mathbb{F})$ such that

$$q(z) = s(z)p(z) + r(z) \tag{1}$$

such that $\deg(r) < \deg(p)$. Prove that this factorization is unique. Hint: to prove uniqueness we assume that there are some other polynomials $s'(z)$ and $r'(z)$ such that $q(z) = s'(z)p(z) + r'(z)$ with $\deg(r') < \deg(p)$ and then prove that this implies that $s(z) = s(z)'$ and $r(z) = r'(z)$ for all $z$.

7. Show that the span of any list of vectors in $\mathbb{V}$ is a subspace of $\mathbb{V}$.

8. Show that span$(v_1, \ldots, v_n)$ is the smallest subspace of $\mathbb{V}$ that contains all the vectors in the list $(v_1, \ldots, v_n)$.

9. (Text 2.1) Prove that if $(v_1, \ldots, v_n)$ spans $\mathbb{V}$ then so does the list

$$(v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_{n-1} - v_n, v_n) \tag{2}$$

10. (Text 2.3) Suppose thet $(v_1, \ldots, v_n)$ is linearly independent in $\mathbb{V}$ and let $w \in \mathbb{V}$. Prove that if $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \text{span}(v_1, \ldots, v_n)$.

11. (Text 2.6) Prove that the real vector space consisting of all continuous real-value functions on the interval $[0, 1]$ is infinite dimensional.

12. (Text 2.8) Let $\mathbb{U}$ be the subspace of $\mathbb{R}^5$ defined by

$$\mathbb{U} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 | x_1 = 3x_2 \wedge x_3 = 7x_4\} \tag{3}$$

Find a basis of $\mathbb{U}$.

13. (Text 2.11) Let $\mathbb{V}$ be a finite dimensional vector space; and let $\mathbb{U}$ be a subspace of $\mathbb{V}$ with $\dim(\mathbb{U}) = \dim(\mathbb{V})$. Prove that $\mathbb{U} = \mathbb{V}$.

14. (Text 2.17) Let $\mathbb{V}$ be finite dimensional. Prove that if $\mathbb{U}_1, \ldots, \mathbb{U}_n$ are subspaces of $\mathbb{V}$ with
$$\mathbb{V} = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n \tag{4}$$
then
$$\dim \mathbb{V} = \dim \mathbb{U}_1 + \cdots + \dim \mathbb{U}_n \tag{5}$$

15. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. Prove that $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is a vector space under addition and scalar multiplication of linear maps.

16. (Text 3.7) Prove that if $(v_1, \ldots, v_n)$ spans $\mathbb{V}$ and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ is surjective (onto), then $(Tv_1, \ldots, Tv_n)$ spans $\mathbb{W}$.

17. (Text 3.11) Let $\mathbb{V}$ and $\mathbb{W}$ be a vector spaces and let $T$ be a linear map, $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ such that $\dim \operatorname{null}(T) < \infty$ and $\dim \operatorname{range}(T) < \infty$ (i.e., the range and null space of $T$ are finite dimensional). Prove that $\dim(\mathbb{V}) < \infty$, i.e., that $\mathbb{V}$ is finite dimensional (Note: Theorem 3.4 in the text does not apply here. Why?)

18. (Text 3.15) Suppose that $\mathbb{V}$ is a finite dimensional vector space, and that $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Prove that $T$ is surjective (onto) if and only if ther exists an $S \in \mathcal{L}(\mathbb{W}, \mathbb{V})$ such that $TS$ is the identity map on $\mathbb{W}$.

19. (Text 3.22) Suppose that $\mathbb{V}$ is finite dimensional and that the $S, T \in \mathcal{L}(\mathbb{V})$. Prove that $ST$ is invertible if and only if both $S$ and $T$ are invertible.

20. (Text 3.23) Let $\mathbb{V}$ be finite dimensional and $T \in \mathcal{L}(\mathbb{V})$. Use the result of the previous exercise to prove that $ST = I \iff TS = I$.

21. (Text 5.5) Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all the eigenvalues and eigenvectors of $T$.

22. (Text 5.11) Suppose that $S, T \in \mathcal{L}(\mathbb{V})$. Show that $ST$ and $TS$ have the same eigenvalues.

23. (Text 5.14) Suppose that $S, T \in \mathcal{L}(\mathbb{V})$, and that $S$ is invertible. Prove that if $p \in \mathbb{P}(\mathbb{F})$ is a polynomial then
$$p(STS^{-1}) = Sp(T)S^{-1} \tag{6}$$

24. (Text 5.15) Suppose that $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(\mathbb{V})$, $p \in \mathbb{P}(\mathbb{C})$, and $a \in \mathbb{C}$. Prove that $a$ is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue of $T$.

25. Let $\mathbb{P}_m(\mathbb{F})$ be the set of polynomials with coefficients in $\mathbb{F}$. Prove that

$$\langle p, q \rangle = \int_0^1 p(x)\overline{q(x)}dx \tag{7}$$

is an inner product.

26. (Text 6.6) Let $\mathbb{V}$ be a real inner product space. Prove that $\langle u, v \rangle = \dfrac{\|u+v\|^2 - \|u-v\|^2}{4}$

27. (Text 6.10). Let $\mathbb{V} = \mathbb{P}_2(\mathbb{R})$ and define the inner product as

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx \tag{8}$$

Let $B = (1, x, x^2)$. Apply the Gram-Schmidt process to $B$ to find an orthonormal basis.

28. (Text 6.24) Using the inner product defined in the previous problem, find a polynomial $q(x) \in \mathbb{P}_2(\mathbb{R})$ such that $p(1/2) = \langle p, q \rangle$ for every $p \in \mathbb{P}_2(\mathbb{R})$.

29. (Text 6.28) Let $T \in \mathcal{L}(\mathbb{V})$ be an operator over $\mathbb{V}$ and let $\lambda \in \mathbb{F}$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\overline{\lambda}$ is an eigenvalue of $T^*$.

30. Let $\mathbb{V} = \mathbb{R}^2$ and define the operator $T \in \mathcal{L}(\mathbb{V})$ as the 90-degree rotation operator given by

$$T(x, y) = (-y, x) \tag{9}$$

where $x, y \in \mathbb{R}$ (i.e, $x, y$ are the components of a vector in $\mathbb{V}$. Show (a) that $T$ is normal and (b) $T$ does not have any eigenvalues in $\mathbb{R}$.

31. Find the polar decomposition of $\begin{pmatrix} 11 & -5 \\ -2 & 10 \end{pmatrix}$

32. For $M = \begin{pmatrix} 11 & -5 \\ -2 & 10 \end{pmatrix}$ (a) Find the singular values of $M$. (b) Find a singular value decomposition of $M$.

33. Verify the Cayley-Hamilton Theorem for

$$M = \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix} \tag{10}$$

34. Let

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \tag{11}$$

Find (a) The eigenvalues; (b) The generalized eigenvectors; (c) The Jordan Form of $M$.