Math 326 Fall 2010

Homework Set Chapter 5

Solutions

5.1.7 Let $A = \{c, d, f, g\}$, $B = \{f, g\}$, and $C = \{d, g\}$.

(a) Is $B \subseteq A$? Yes, because every element of B is in A.
(b) Is $C \subseteq A$? Yes, because every element of C is in A.
(c) Is $C \subseteq C$? Yes, because every element of C is in C.
(d) is C a proper subset of C? No, because there are no elements of C that are not in C.

5.1.8 (a) Is $3 \in \{1, 2, 3\}$? Yes
(b) Is $1 \in \{1\}$? Yes
(c) Is $\{2\} \in \{1, 2\}$? No
(d) Is $\{3\} \in \{1, \{2\}, \{3\}\}$? Yes
(e) Is $1 \in \{1\}$? Yes
(f) Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$? No, because $2 \notin$ the set.
(g) Is $\{1\} \subseteq \{1, 2\}$? Yes
(h) Is $1 \in \{\{1\}, 2\}$? No
(i) Is $\{1\} \subseteq \{1, \{2\}\}$? Yes
(j) Is $\{1\} \subseteq \{1\}$? Yes

5.1.11 Let $A = \{x \in \mathbb{R} | -3 \leq x \leq 0\}$,$B = \{x \in \mathbb{R} | -1 < x < 2\}$, $C = \{x \in \mathbb{R} | 6 < x \leq 8\}$.

(a) $A \cup B = \{x \in \mathbb{R} | -3 \leq x < 2\}$
(b) $A \cap B = \{x \in \mathbb{R} | -1 < x \leq 0\}$
(c) $A^C = \{x \in \mathbb{R} | x < -3 \vee x > 0\}$
(d) $A \cup C = \{x \in \mathbb{R} | -3 \leq x \leq 0 \vee 6 < x \leq 8\}$
(e) $A \cap C = \emptyset$
(f) $B^C = \{x \in \mathbb{R} | x \leq -1 \vee x \geq 2\}$
(g) $A^C \cap B^C = \{x \in \mathbb{R} | x < -3 \vee x \geq 2\}$
(h) $A^C \cup B^C = \{x \in \mathbb{R} | x \leq -1 \vee x > 0\}$
(i) $(A \cap B)^C = \{x \in \mathbb{R} | x \leq -1 \vee x > 0\}$
(j) $(A \cup B)^C = \{x \in \mathbb{R} | x < -3 \vee x \geq 2\}$

5.1.12 True or False

(a) $\mathbb{Z}^+ \subset \mathbb{Q}$ True
(b) $\mathbb{R}^- \subset \mathbb{Q}$ False
(c) $\mathbb{Q} \subset \mathbb{Z}$ False
(d) $\mathbb{Z}^- \cup \mathbb{Z}^+ = \mathbb{Z}$ False (missing 0)
(e) $\mathbb{Z}^- \cap \mathbb{Z}^+ = \emptyset$ True
(f) $Q \cap \mathbb{R} = \mathbb{Q}$ True
(g) $Q \cup \mathbb{Z} = \mathbb{Q}$ True
(h) $\mathbb{Z}^+ \cap \mathbb{R} = \mathbb{Z}^+$ True
(i) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Z}$ False

5.1.29 Let $A = \{x, y, z, w\}$ and $B = \{a, b\}$.

(a) $A \times B = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b), (w, a), (w, b)\}$

(b) $A \times A = \{(x, x), (x, y), (x, z), (x, w), (y, x), (y, y), (y, z), (y, w), (z, x), (z, y), (z, z), (z, w), (w, x), (w, y), (w, z), (w, w)\}$

(c) $B \times A = \{(a, x), (a, y), (a, z), (a, w), (b, x), (b, y), (b, z), (b, w)\}$

(d) $B \times B = \{(a, a), (a, b), (b, a), (b, b)\}$

5.1.32 Trace table:

i	1			2					3
j	1	2	3	1	2	3	4	5	
found	no	yes		no					
answer	$A \subseteq B$							$A \not\subseteq B$	

5.1.33 Algorithm:

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Input: a[1], a[2], . . . , a[n] [a one-dimensional array],
x [an element of the same data type as
the elements of the array]

Algorithm Body:
  i = 1
  answer = "x \not \in A"
  while (i <= n and answer = "x \not \in A")
    if x = a[i] then answer := "x \in A"
    i = i + 1
 end while

Output: answer [a string]
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5.2.2 (a) Suppose A and B are sets. To show that $A - B \subseteq A$ we must show that every element in $A - B$ is in A. But any element in $A - B$ is in A and not in B by the definition of A-B. In particular, such an element is in A.

(b) Suppose that A and B are sets and $x \in A - B$. We must show that $x \in A$. By the definition of set difference, $x \in A$ and $x \notin B$. In particular, $x \in A$, which is what was to be shown.

5.2.3 Suppose that A, B, C are sets and that $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$ we must show that every element of A is in C. By given any element of A, that element is in B (because $A \subseteq B$), so that element is also in C (because $B \subseteq C$). Hence $A \subseteq C$.

5.2.4 Suppose that A and B are any sets and $A \subseteq B$. We must show that $A \cup B \subseteq B$. Let $x \in A \cup B$. We must show that $x \in B$. By definition of union, $x \in A$ and $x \in B$. In the case $x \in A$, then since $\subseteq B$, $x \in B$. In the case $x \in B$, then clearly $x \in B$. So in either case $x \in B$, as was to be shown.

5.2.8 Use an element argument to prove that

$$(A - B) \cup (C - B) = (A \cup C) - B$$

Let $x \in (A - B) \cup (C - B)$.

Either $x \in (A - B)$ or $x \in (C - B)$.

If $x \in A - B$ then $x \in A \cup C$ and $x \notin B$. Hence by definition of set difference $x \in (A \cup C) - B$.

If $x \in C - B$ then $x \in A \cup C$ and $x \notin B$. Hence by definition of set difference $x \in (A \cup C) - B$.

Hence by generality $(A - B) \cup (C - B) \subseteq (A \cup C) - B$

Now let $x \in (A \cup C) - B$.

By definition of set difference $x \notin B$.

Also by definition of set difference $x \in A \cup C$.

By definition of union, either $x \in A$ or $x \in C$.

If $x \in A$ then since $x \notin B$, $x \in A - B$.

If $x \in C$ then since $x \notin C$, $x \in C - B$.

Since either $x \in A - B$ or $x \in C - B$ by definition of union, $x \in (A - B) \cup (C - B)$.

By generality, $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

Since we have already show $(A - B) \cup (C - B) \subseteq (A \cup C) - B$ we can conclude that $(A - B) \cup (C - B) = (A \cup C) - B$.

5.2.10 Use an element argument to prove that for all sets A and B, $A \cup (A \cap B) = A$.

Let $x \in A \cup (A \cap B)$. Then by definition of union, either $x \in A$ or $x \in A \cap B$. By the first statement $x \in A$, and generality, we conclude that $A \cup (A \cap B) \subseteq A$.

Let $x \in A$. Then by definition of Union, $x \in A \cup (\text{anything})$ and in particular, $x \in A \cup (A \cap B)$. Hence by generality, $A \subseteq A \cup (A \cap B)$.

Hence $A = A \cup (A \cap B)$.

5.2.14 Use an element argument to prove that if $A \subseteq B$ then $B^C \subseteq A^C$.

Let $x \in B^C$. Then by definition of Complement $x \notin B$.

Suppose $A \subseteq B$. Either $x \in A$ or $x \notin A$.

If $x \in A$ then by definition of subset, $x \in B$. But we know that $x \not inB$. Hence this is not possible. Hence it must be that $x \notin A$.

Hence $x \in A^C$. Since x was completely general, by generality we can extend this to any element of B^C to conclude that every element of B^C is in A^C. Hence $B^C \subseteq A^C$, as required.

5.2.17 Use an element argument to show that for all sets A, B, C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C) \tag{1}$$

Let $x \in A \times (B \cap C)$. Then $x = (a, q)$ where $a \in A$ and $q \in B \cap C$.

Since $q \in B \cap C$, then $q = b$ for some $b \in B$ and $q = c$ for some $c \in C$.

Hence $x = (a, b)$ and $x = (a, c)$.

Since $x = (a, b)$ for some $b \in B$, $x \in A \times B$.

Since $x = (a, c)$ for some $c \in C$, $x \in A \times C$.

Since $x \in A \times B$ and $x \in A \times C$, by the definition of intersection, $x \in (A \times B) \cap (A \times C)$.

By generalization $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Now let $x \in (A \times B) \cap (A \times C)$.

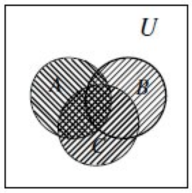
Then for some $a \in A$, some $b \in B$, and some $c \in C$, $x = (a, b)$ and $x = (a, c)$.

Hence $x = (a, q)$ where $q \in B$ and $q \in C$. Hence $q \in B \cap C$. Hence $x \in A \times (B \cap C)$.

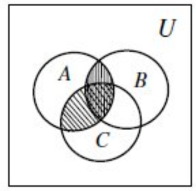
Hence by generality $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

21. *b*

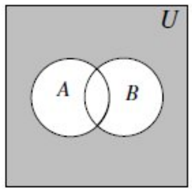


darkly shaded region is $A \cap (B \cup C)$

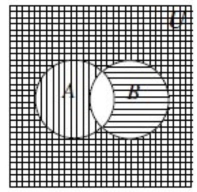


entire shaded region is $(A \cap B) \cup (A \cap C)$

c

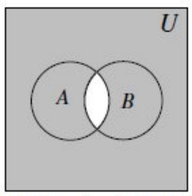


shaded region is $(A \cup B)^c$

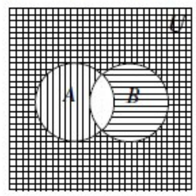


cross-hatched region is $A^c \cap B^c$

d



shaded region is $(A \cap B)^c$



entire shaded region is $A^c \cup B^c$

5.2.21

5.2.29 For all sets A and B prove that if $B \subseteq A^C$ then $A \cap B = \emptyset$.

Let A and B be sets with $B \subseteq A^C$.

Suppose $A \cap B = \emptyset$ i.e., , suppose there is an element x in $A \cap B$.

Then by definition of intersection, $x \in A$ and $x \in B$, But $B \subseteq A^C$ and so by definition of subset, $x \in A^C$.

By definition of complement this means that $x \notin A$.

Hence $x \in A$ and $x \notin A$, which is a contradiction. Thus the proposition is false, and we conclude that $A \cap B = \emptyset$.

5.2.30 For all sets A, B, and C, if $A \subseteq B$, and $B \cap C\emptyset$ then $A \cap C = \emptyset$.

Let A, B, and C be any sets such that $A \subseteq B$ and $B \int C = \emptyset$.

Suppose $A \cap C \neq \emptyset$.

Then there is some element x in $A \cap C$.

By definition of intersection, $x \in A$ and $x \in C$.

But by hypothesis $A \subseteq B$ and since $x \in A$ then $x \in B$.

Hence $x \in B \cap C$.

Hence $B \cap C \neq \emptyset$. This is a contradiction so the assumption must be false.

5.3.2 Find a counterexample to to show that the following is false: For all sets A, B and C, if $A \subseteq B$ then $A \cap (B \cap C)^C = \emptyset$.

Universe $= \{1, 2, 3, 4\}$

$A = \{1, 2\}$

$B = \{1, 2, 3\}$

$C = \{2\}$

$$\begin{aligned} A \cap (B \cap C)^C &= \{1,2\} \cap (\{1,2,3\} \cap \{2\})^C \\ &= \{1,2\} \cap (\{2\})^C \\ &= \{1,2\} \cap (\{1,3,4\} \\ &= \{1\} \neq \emptyset \end{aligned}$$

5.3.10 For all sets A, B, C, If $A \subseteq B$ then $A \cap B^C = \emptyset$.

Proof. Let A and B be sets such that $A \subseteq B$.

Suppose that $A \cap B^C \neq \emptyset$.

Then $\exists x \in A \cap B^C$.

By definition of intersection, $x \in A$ and $x \in B^C$.

By definition of complement $x \notin B$.

But $A \subseteq B$, so since $x \in A$, then $x \in B$.

This is a contradiction.

5.3.12 For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

This is false.

Counterexample: $A = \{1, 2\}$, $B = \{3\}$.

Then $A \cap B = \emptyset$.

$A \times B = \{(1, 3), (2, 3)\} \neq \emptyset$.

5.3.27 For all sets A, B, C, $(A - B) - C = A - (B \cup C)$

$$\begin{aligned} (A - B) - C &= (A - B) \cap C^C && \text{set difference law} \\ &= (A \cap B^C) \cap C^C && \text{set difference law} \\ &= A \cap (B^C \cap C^C) && \text{associative law} \\ &= A \cap (B \cup C)^C && \text{De Morgan} \\ &= A - (B \cup C) && \text{Set Difference Law} \end{aligned}$$

5.3.30 For all sets A and B, $(B^C \cup (B^C - A))^C = B$

$$\begin{aligned} (B^C \cup (B^C - A))^C &= (B^C \cup (B^C \cap A^C))^C && \text{Set Difference Law} \\ &= (B^C)^C \cap (B^C \cap A^C)^C && \text{De Morgan} \\ &= B \cap ((B^C)^C \cup (A^C)^C) && \text{Double Comp and DeMorgan (2nd Term)} \\ &= B \cap (B \cup A) && \text{Double Complement} \\ &= B && \text{Absorption} \end{aligned}$$

5.3.56 (skip)