Math 326 Fall 2010

Homework Set 1

Solutions

2.1-4 Let $Q(n)$ be the predicate "$n^2 \leq 30$."

(a) Write $Q(2)$, $Q(-2)$, $Q(7)$, and $Q(-7)$ and indicate which of these statements are true and which are false.
$Q(2) \iff 2^2 \leq 30 \iff 4 \leq 30 \iff T$
$Q(-2) \iff (-2)^2 \leq 30 \iff 4 \leq 30 \iff T$
$Q(7) \iff 7^2 \leq 30 \iff 49 \leq 30 \iff F$
$Q(-2) \iff (-7)^2 \leq 30 \iff 49 \leq 30 \iff F$

(b) Find the truth set of $Q(n)$ if the domain of n is $\mathbb{Z}$.
To find the truth set we must solve $n^2 \leq 30$ for all values of n that are integers, i.e., $-\sqrt{30} \leq n \leq \sqrt{30}$. The truth set is thus

$$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

(c) If the domain is the set $\mathbb{Z}^+$ of all positive integers, what is the truth set of $Q(n)$?
To find the truth set we must solve $n^2 \leq 30$ for all values of n that are positive integers, i.e., $0 < n \leq \sqrt{30}$. The truth set is thus

$$\{1, 2, 3, 4, 5\}$$

2.1-5 Let $Q(x, y)$ be the predicate "If $x < y$ then $x^2 < y^2$" with domain for both x and y being the set $\mathbb{R}$ of real numbers.

(a) Explain why $Q(x, y)$ is false if $x = -2$ and $y = 1$.
$Q(-2, 1)$ says $(-2 < 1) \implies (4 < 1)$, which is $T \implies F$, which is F.

(b) Give values different from those in part (a) for which $Q(x, y)$ is false.
Try $x = -1$ and $y = 0$. Then $Q(-1, 0)$ says $-1 < 0 \implies 1 < 0$, which is also $T \implies F$, and hence F.

(c) Explain why $Q(x, y)$ is true if $x = 3$ and $y = 8$.
$Q(3, 8)$ says $(3 < 8) \implies (9 < 64)$ which is $T \implies T$ which is T.

(d) Give values different from those in part (c) for which $Q(x, y)$ is true.
Try $x = 2$ and $y = -3$. Then $Q(2, -3)$ says $(2 < -3) \implies (4 < 9)$ which is $F \implies T$ which is T.

2.1-7 Find the truth of each predicate.

(a) predicate: $6/d$ is an integer, domain $\mathbb{Z}$.

$$\{-6, -3, -2, -1, 1, 2, 3, 6\}$$

(b) predicate: $6/d$ is an integer, domain $\mathbb{Z}^+$.

$$\{1, 2, 3, 6\}$$

(c) predicate: $1 \leq x^2 \leq 4$, domain $\mathbb{R}$.

$$\{x \in \mathbb{R} | -2 \leq x \leq -1 \vee 1 \leq x \leq 2\}$$

(d) predicate: $1 \leq x^2 \leq 4$, domain $\mathbb{Z}$.

$$\{-2, -1, 1, 2\}$$

2.1-12 (Find counterexamples to show that the statement is false): $\forall$ real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

Try $x = y = 1$. Then $\sqrt{x+y} = \sqrt{1+1} = \sqrt{2} \neq 2 = 1+1 = \sqrt{1} + \sqrt{1} = \sqrt{x} + \sqrt{y}$

2.1-16 Rewrite each of the following statements in the form "$\forall$______ x, ______."

(a) All dinosaurs are extinct.

$$\forall \text{ dinosaur } x, x \text{ is extinct}$$

(b) Every real number is positive, negative, or zero.

$$(\forall x \in \mathbb{R})((x > 0) \vee (x < 0) \vee (x = 0))$$

(c) No irrational numbers are integers.

$$\forall \text{ irrational number } x, x \notin \mathbb{Z}$$

(d) No logicians are lazy.

$$\forall \text{ logicians } x, x \text{ is not lazy}$$

(e) The number 2,147,581,953 is not equal to the square of any integer.

$$\forall m \in \mathbb{Z}, m^2 \neq 2,147,581,953$$

(f) The number -1 is not equal to the square of any real number.

$$\forall x \in \mathbb{R}, x^2 \neq -1$$

2.1-21 Rewrite each of the following statements in the form "$\forall$______ x, if ______ then ______" or "$\forall$______ x and y, if ______ then ______."

(a) All Java programs have at least 5 lines.

$$\forall x, \text{ if } x \text{ is a Java program, then } x \text{ has at least 5 lines}$$

(b) Any valid argument with true premises has a valid conclusion.
$\forall$ arguments x, if x is valid $\wedge\, x$ has true premises then x has a true conclusion
OR: $\forall x$, if x is a valid argument with true premises, then x has a true conclusion
OR: $\forall$ valid arguments x, if x has true premises, then x has a true conclusion

(c) The sum of any two even integers is even.

$$\forall \text{ integers } x \text{ and } y, \text{ if } x \text{ and } y \text{ are even, then } x+y \text{ is even}$$

(d) The product of any two odd integers is odd.

$$\forall \text{ integers } x \text{ and } y, \text{ if } x \text{ and } y \text{ are odd, then } xy \text{ is odd}$$

2.1-22 Rewrite eqch of the following statements in the two forms form "$\forall$_____ x, if _____ then _____" and "$\forall$_____ x, _____" (without the if-then).

(a) The square of any even integer is even.

$$\forall x, \text{ if } x \text{ is an even integer, then } x^2 \text{ is even}$$

$$\forall \text{ even integers} x, x^2 \text{ is even}$$

(b) Every computer science student needs to take data structures.

$$\forall x, \text{ if } x \text{ is a computer science student, then } x \text{ needs to take data structures}$$

$$\forall \text{ computer science students } x, x \text{ needs to take data structures}$$

2.2-1 Which of the following is a negation for "All discrete mathematics students are atheletic." More than one answer may be correct.

(a) **[There is a discrete math student who is nonathletic.]**
(b) All discrete mathematics students are nonathletic.
(c) There is an athletic person who is a discrete math student.
(d) No discrete math students are athletic.
(e) **[Some discrete math students are nonathletic.]**
(f) Some nonathletic people are not discrete math students.

2.2-2 Which of the following is a negation for "All dogs are loyal?"

(a) All dogs are disloyal.
(b) No dogs are loyal.
(c) **[Some dogs are disloyal.]**
(d) Some dogs are loyal.
(e) There is a disloyal animal that is a dog.
(f) **[There is a dog that is disloyal.]**
(g) No animals that are not dogs are loyal.
(h) Some animals that are not dogs are loyal.

2.2-3 Write a formal negation for each of the following statements.

(a) $\forall$ fish x, x has gills.
$\exists$ fish x, x has no gills

(b) $\forall$ computers c, c has a CPU.
$\exists$ computer c, c has no CPU

(c) $\exists$ a movie m such that m is over 6 hours long.
$\forall$ movies m, m is less than or equal to 6 hours long.

(d) $\exists$ a band b such that b has won at least 10 grammy awards
$\forall$ bands b, b has won fewer than 10 grammy awards

2.2-8 Consider the statement "There are no simple solutions to life's problems." Write an informal negation for the statement, and then write the statement formally using quantifiers and variables.

Informal Negation: There are some simple solutions to life's problems.

Formal versions of statement:
$\forall$ solutions to life's problems x, x is not simple.
OR: $\forall x, (x \text{ is a solution}) \implies (x \text{ is not simple})$

2.2-9 Write a negation to: $\forall x \in \mathbb{R}, x > 3 \implies x^2 > 9$

$(\exists x \in \mathbb{R})(x \leq 3) \wedge (x^2 \leq 9)$

2.2-14 Determine if the proposed negation is correct. If it is not correct, write a correct negation.

Statement: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 = x_2$.

Proposed negation: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 \neq x_2$.

The proposed negation is not correct. The correct negation of $(\forall x)(P(x) \implies Q(x))$ is $(\exists x)(P(x) \wedge \sim Q(x))$. Thus the correct negation is this:

$(\exists x_1, x_2 \in \mathbb{R})(x_1^2 = x_2^2 \wedge x_1 \neq x_2)$

2.2-28 True or False? All the occurrences of the letter u in the title of this book are lower case. Justify your answer.

Write the statement formally: $(\forall u \in$ the book's title$)(u$ is in lower case$)$.

It's negation is: $(\exists U \in$ the book's title$)(U$ is in upper case$)$.

But the book's title is "Discrete Mathematic's with Applications." There are not any upper case U's in the title. Hence the negation is F. Since it's negation is F, the original statement must be true.

Another way to look at this is that this is of the form $(\forall x)(P(x) \implies Q(x))$ where $P(x)$ is F because there are no u's of any kind in the title, hence we have a vacuously true statement.

2.3-10 This exercise refers to example 2.3.3, Determine whether each of the following statements is True or False.

(a) $\forall$ students $S, \exists$ a dessert D such that S chose D

True. Every student chose at least one dessert. Uta chose pie, Tim chose both pie and cake, and Yuen chose pie.

(b) $\forall$ students $S, \exists$ a salad T such that S chose T
False, because Yuen did not chose a salad.

(c) $\exists$ a dessert D, such that $\forall$ students S, S chose D
True, because everybody chose pie.

(d) $\exists$ a beverage B, such that $\forall$ students S, S chose B
False, because if there was a beverage that everyone chose there would be a beverage in the beverage box that had lines connecting it to all three names.

(e) $\exists$ an item I, such that $\forall$ students S, S did not choose I
False, because everything was chosen by somebody.

(f) $\exists$ a station Z such that $\forall$ students $S, \exists$ an item I such that S chose I from Z.
True - in fact there are three solutions: beverage, main course, and dessert.

2.3-13 Let $D = E = \{-2, -1, 0, 1, 2\}$. Write negations for each of the following statements and determine which is true, the given statement or its negation.

(a) $(\forall x \in D)(\exists y \in E)(x + y = 1)$
Negation: $(\exists x \in D)(\forall y \in E)(x + y \neq 1)$
The negation is true because $x = -2 \in D$, and the only way $x + y = 1$ is if $y = 3$, but $y = 3 \notin E$.

(b) $(\exists x \in D)(\forall y \in E)(x + y = -y)$
Negation:$(\forall x \in D)(\exists y \in E)(x + y \neq -y)$
The negation is true.
If $x = -2$, then $y = 2$ gives $x + y = 0 \neq -2 = -y$.
If $x = -1$, then $y = 1$ gives $x + y = 0 \neq -1 = -y$.
If $x = 0$, then $y = 1$ gives $x + y = 1 \neq -1 = -y$.
If $x = 1$, then $y = 1$ gives $x + y = 2 \neq -1 = -y$.
If $x = 2$, then $y = 2$ gives $x + y = 4 \neq -4 = -y$.

2.3-23 Use the laws for negating universal and existential statements to derive the following rules.

(a) $\sim (\forall x \in D(\forall y \in E(P(x, y)))) \equiv \exists x \in D(\exists y \in E(\sim P(x, y)))$

$$\begin{aligned} \sim (\forall x \in D(\forall y \in E(P(x, y)))) &\equiv \exists x \in D(\sim (\forall y \in E(P(x, y)))) \\ &\equiv \exists x \in D(\exists y \in E(\sim P(x, y))) \end{aligned}$$

(b) $\sim (\exists x \in D(\exists y \in E(P(x, y)))) \equiv \forall x \in D(\forall y \in E(\sim P(x, y)))$

$$\begin{aligned} \sim (\exists x \in D(\exists y \in E(P(x, y)))) &\equiv \forall x \in D(\sim (\exists y \in E(P(x, y)))) \\ &\equiv \forall x \in D(\forall y \in E(\sim P(x, y))) \end{aligned}$$

2.3-30 Consider the statement "Everybody is older than somebody." Rewrite the statement in the form "$\forall$ people x, $\exists$ ______."

$\forall$ people x, $\exists$ a person y such that x is older than y.

2.3-31 Consider the statement "Somebody is older than everybody." Rewrite the statement in the form "$\exists$ a person x such that $\forall$ ______."

$\exists$ a person x such that $\forall$ people y, x is older than y.

2.3-42 Write a negation for the following statement, which is the definition of $\lim_{x \to a} f(x) = f(a)$.

"For all real numbers $x > 0$, there exists a real number $\delta > 0$, such that for all real numbers x, if $a - \delta < x < a + \delta$, then $f(a) - \epsilon < f(x) < f(a) + \epsilon$."

The negation is: $\exists$ a real number $\epsilon > 0$ such that $\forall$ real numbers $\delta > 0$, $\exists$ a real number x such that $a - \delta < x < a + \delta$ and either $L - \epsilon \geq f(x)$ or $f(x) \geq L + \epsilon$.

2.4-3 Use universal instantiation or universal modus ponens to fill in the valid conclusions for the following:

For all real number a, b, c and d, if $b \neq 0$ and $d \neq 0$, then $a/b + c/d = (ad + bc)/bd$.

$a = 2, b = 3, c = 4$ and $d = 5$ are particular real numbers such that $b \neq 0$ and $d \neq 0$.

$$\boxed{\therefore \frac{2}{3} + 45 = \frac{(2)(5) + (3)(4)}{(3)(5)} \left(= \frac{22}{15} \right)}$$

2.4-4 Use universal instantiation or universal modus ponens to fill in the valid conclusions for the following:

For all real numbers r, a, and b, if r is positive, then $(r^a)^b = r^{ab}$.

$r = 3$, $a = 1/2$, and $b = 6$ are particular real numbers such that r is positive.

$$\boxed{\therefore (3^{1/2})^6 = 3^{(1/2)(6)} (= 3^3 = 27)}$$

2.4-6 Use universal modus tollens to fill in valid conclusions for the argument:

If a computer program is correct, then the compilation of the program does not produce any error mesages.

Compilation of this program produces error messages.

$\boxed{\therefore \text{ this computer program is not correct.}}$

2.4-11 State whether the following is valid or invalid and justify.

All cheaters sit in the back row.

Monty sits in the back row.

Therefore Monty is a cheater.

Invalid - converse error. The steps can be written as $P \implies Q$; Q; $Q \implies P$. The third statement is the converse of the first which is not necessarily true.

2.4-12 State whether the following is valid or invalid and justify.

All honest people pay their taxes.

Darth is not honest.

Thus Darth does not pay his taxes.

This is invalid by inverse error. The statements are $P \implies Q$; $\sim P$; therefore $\sim Q$, which assumes that $\sim P \implies \sim Q$, which is not necessarily true.

2.4-13 State whether the following is valid or invalid and justify.

For all students x, if x studies discrete math, then x is good at logic.

Tarik studies discrete math.

Tarik is good at logic.

This is a valid application of universal modus ponens.

2.4-17 State whether the following is valid or invalid and justify.

If an infinite series converges, then the terms go to zero.

The terms of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ go to zero.

Hence the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

This is an invalid use of the converse.

2.4-18 State whether the following is valid or invalid and justify.

If an infinite series converges, then its terms go to zero.

The terms of the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ do not go to zero.

Thus the infinite series doe s not converge.

This is a valid application of universal modus tollens.

2.4-19 Rewrite the statement "No good cars are cheap" in the form "$\forall all x$, if $P(x)$ then $\sim Q(x)$.". Indicate whether each of the following argument sis valid or invalid and justify your answers.

$\forall x$, if x is a good car, then x is not cheap.

(a) No good car is cheap; A Rimbaud is a good car; Hence A Rimbaud is not cheap.
Valid universal Modus ponens.

(b) No good car is cheap; A Simbaru is not cheap; Hence a Simbaru is a good car.
Invalid: converse error.

(c) No good car is cheap; A VX roadster is cheap; A VX roadster is not good.
Valid: universal modus tollens.

(d) No good car is cheap; An Omnex is not a good car; An omnex is cheap.
Invalid: Inverse error.

3.1-3 Assume that r and s are particular integers.

(a) Is $4rs$ even?
Yes, because $4rs = 2(2rs)$ and $2rs \in \mathbb{Z}$.

(b) Is $6r + 4s^2 + 3$ odd?
Yes, because $6r + 4s^2 + 3 = 2(3r + 2s^2 + 1) + 1$ and $3r + 2s^2 + 1 \in \mathbb{Z}$.

(c) If r and s are both positive, is $r^2 + 2rs + s^2$ composite?
Yes, because $r^2 + 2rs + s^2 = (r + s)^2$, i.e., the product of two numbers each of which is greater than or equal to 2, hence neither of them is equal to one.

3.1-6 Prove that there are real numbers a and b such that

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

Let $a = 1$ and $b = 0$. Then $\sqrt{a+b} = 1$ and $\sqrt{a} + \sqrt{b} = 1 + 0 = 1$.

3.1-8 Prove that there is a real number x such that $x > 1$ and $2^x > x^{10}$.

An example is given by $x = 80$.

$2^{80} = (2^{10})^8 = (1024)^8 = (1024)^3(1024)^5 > (1024)^3(1000)^5 = (1024)^3(10^3)^5 = (1024)^3(10^{15})$

$80^{10} = (8 \times 10)^{10} = (2^3)^{10}(10)^{10} = (2^{10})^3(10)^{10} = (1024)^3(10)^{10} < (1024)^3(10)^{15} < 2^{80}$

No calculator needed.

3.1-12 Disprove by counterexample: For all integers n, if n is odd, then $(n-1)/2$ is odd.

Let $n = 9$. Then $n - 1 = 9 - 1 = 8$. Hence $(n-1)/2 = 8/2 = 4$ which is even.

3.1-16 Disprove by counterexample: The average of any two odd integers is odd.

For example, the average of 1 and 3 is 2, which is not odd, providing a counter-example.

3.1-36 Find the mistake in the proof.

Theorem: For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

Invalid Proof: Suppose k is any integer such that $k > 0$. If $k^2 + 2k + 1$ is composite, then $k^2+2k+1 = rs$ for some integers r and s such that $1 < r < (k^2+2k+1)$ and $1 < s < k^2+2k+1$. Since $k^2+2k+1 = rs$, and both r and s are strictly between q and k^2+2k+1, then k^2+2k+1 is not prime. Hence $k^2 + 2k + 1$ is composite.

The word Since in the box is totally unjustified - the previous sentence does not tell us anything about $k^2 + 2k + 1$ because it is an if-then statement, for which the premise is the thing we are trying to prove.

3.1-37 Find the mistake in the proof.

Theorem: The product of an even integer and an odd integer is even.

Invalid Proof: Suppose m is an even integer and n is an odd integer. If mn is even, then by definition of even , there exits an integer r such that $mn = 2r$. Also, since m is even, there exists an integer p such that $m = 2p$, and since n is odd , there exists an integer q such that $n = 2q + 1$. Thus

$$mn = (2p)(2q + 1) = 2r$$

where r is an integer. By definition of even, then mn is even.

The second sentence states a conclusion that follows from the assumption that mn is even. The next-to-last sentence states this conclusion as if it were known to be true. But it is not known to be true. In fact, it is the main task of a genuine proof to derive this conclusion, not from the assumption that it is true but from the hypothesis of the theorem.

3.1-38 Find the mistake in the proof.

Theorem: The sum of any two even integers equals $4k$ for some integer k.

Invalid Proof: Suppose that m and n are any two even integers. By definition of even, $m = 2k$ for some integer k, and $n = 2k$ for some some integer k. Hence by substitution, $m + n = 2k + 2k = 4k$. This is what was to be shown.

The mistake in the "proof" is that the same symbol, k, is used to represent two different quantities. By setting both m and n equal to $2k$, the proof specifies that $m = n$, and, therefore, it only deduces the conclusion in case $m = n$. If $m \neq n$, the conclusion is often false. For instance, $6 + 4 = 10$ but $10 \neq 4k$ for any integer k.

3.1-42 The product of any even integer and any even integer is even.

Proof. Let $m, n \in \mathbb{Z}$ such that m is even.

By the definition of even, $\exists k \in \mathbb{Z}$ such that $m = 2k$.

Hence $mn = 2nk = 2(nk)$ (substitution and associativity). Since $nk \in \mathbb{Z}$ (closure of $\mathbb{Z}$ under multiplication), mn is even. □

3.2-5 $x = 0.565656\ldots.$

$100x = 56.5656\ldots$

$99x = 56$ hence $x = 56/99$.

3.2-7 $x = 52.4672167216721...$

$10x = 524.67216721...$

$100,000x = 5,246,721.67216721...$

$99,990x = 5,246,721 - 524 = 5,246,197$ hence $x = 5,246,197/99,990$

3.2-15 Prove or disprove: The difference of any two rational numbers is a rational number.

Proof: Let $x, y \in \mathbb{Q}$. Then by definition of rational numbers there exists integers a, b, c, d, with $b, d \neq 0$, such that

$$x = \frac{a}{b}, y = \frac{c}{d}$$

Hence

$$x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

The numerator is an integer by closure of $\mathbb{Z}$ under multiplication and addition.

The denominator is an integer by closure of $\mathbb{Z}$ under multiplication.

The denominator is nonzero because $b \neq 0$ and $d \neq 0$. Hence $\exists p, q \in \mathbb{Z}$ such that $x - y = p/q$ and $q \neq 0$. Hence $x - y \in \mathbb{Q}$.

3.2-21 True or False: If a is any odd integer, then $a^2 + a$ is even.

Let a be odd. Then $\exists k \in \mathbb{Z}$ such that $a = 2k + 1$. Hence

$$a^2 + a = (2k + 1)^2 + (2k + 1) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$$

3.2-24 Show that if r is rational, so is $3r^2 - 2r + 4$.

Let $r \in \mathbb{Q}$.

By exercise 13, $r^2 \in \mathbb{Q}$. By theorem 3.2.1, $3 \in \mathbb{Q}$. Hence by exercise 13 again, $3r^2 \in \mathbb{Q}$.

By theorem 3.2.1, $-2 \in \mathbb{Q}$. By ex. 13, $-2r \in \mathbb{Q}$. By theorem 3.2.2, the sum of two rationals is rational, hence $3r^2 - 2r \in \mathbb{Q}$.

By theorem 3.2.1, $\in \mathbb{Z}$. By theorem 3.2.2, the sum of two rationals is rational, hence $3r^2 - 2r + 4 \in \mathbb{Q}$.

3.2-33 Find the mistake in the proof that the sum of any two rationals is rational.

Bad Proof: Any two rational numbers produce a rational number when added together. So if r and s are particular but arbitrary chosen rational numbers, then $r + s$ is rational.

This proof assumes the result!

3.2-34 Find the mistake in the proof that the sum of any two rationals is rational.

Bad Proof: Suppose r and s are rational numbers. By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$ and $s = a/b$ for some integers a and b with $b \neq 0$. Then $r+s = a/b+a/b = 2a/b$. Let $p = 2a$. Then p is an integer since it is the product of integers. Hence $r+s = p/b$ where p and b are integers and $b \neq 0$. Thus $r+s$ is rational by the definition of rational. This is what was to be shown.

By setting both r adn s equal to a/b this proof violates the requirement that r and s be arbitrarily chosen rational numbers; the assumption is only true if $r = s$.

3.2-36 Find the mistake in the proof that the sum of any two rationals is rational.

Bad Proof: Suppose r and s are rational numbers. If $r + s$ is rational, then by definition of rational, $r + s = a/b$ for some integers a and b with $b \neq 0$. Also, since r and s are rational, $r = i/j$ and $s = m/n$ for some integers $i, j.m, n$ with $j/neq0$ and $n \neq 0$. It follows that $r + s = i/j + m/n = a/b$ which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown.

The second sentence asserts that a certain conclusionfollows if $r + s$ is rational, and the rest of the proof uses that conclusion to deduce that $r + s$ is rational. Thus this incorrect proof assumes what is to be proved.