Worksheet 6

For each function, (a) Find all aysmptotes; (b) Identify the critical points; (c) Identify possible inflection points; (d) Identify where the function is increasing and decreasing; (e) Find the intervals where the function is concave up or concave down; (f) Find the local maxima/minima; (g) Sketch the function (h) Identify all intercepts

1. $y = \dfrac{x^2-4}{x^2-2x}$

First we simplify:

$$y = \frac{x^2-4}{x^2-2x} = \frac{(x-2)(x+2)}{x(x-2)} = \frac{x+2}{x} = 1 + \frac{2}{x}$$

except at $x = 2$, where the function is undefined. The point $x = 2$ is a removable singularity.

There is a vertical asymptote at $x = 0$.

There is a horizontal asymptote at

$$y = \lim_{x\to\infty}\left(1 + \frac{2}{x}\right) = 1$$

Similarly,

$$y = \lim_{x\to-\infty}\left(1 + \frac{2}{x}\right) = 1$$

Differentiating,

$$y' = -\frac{2}{x^2}, \qquad y'' = \frac{4}{x^3}$$

so the function approaches the same asymptote in both directions.

To see if the function ever crosses the horizontal asymptote set $h = f(x)$, where h is the $y-$value of the asymptote, and solve for x

$$1 = f(x) = 1 + \frac{2}{x} \implies 0 = \frac{2}{x}$$

for which there is no solution. Hence the function never crosses the asymptote.

There is no solution to $y' = 0$ so there are no critical points except for the removable singularity at $x = 2$.

Since $y' = -2/x^2 < 0$ for all x, the function is always decreasing. The intervals are:

$(-\infty, 0)$: decreasing

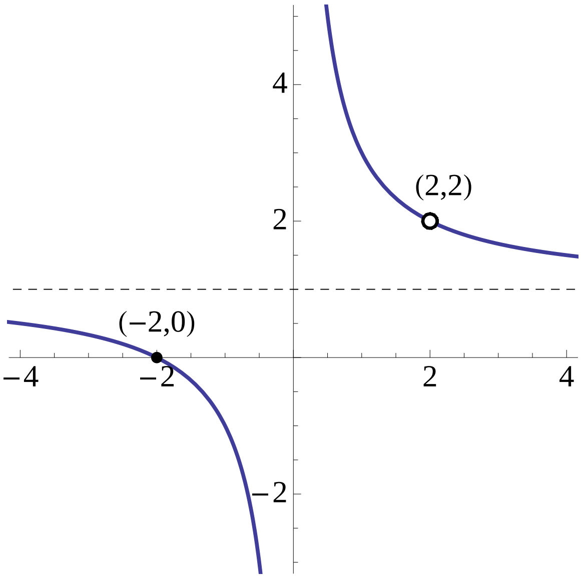
$(0, 2)$: decreasing

$(2, \infty)$: decreasing

There is no solution to $y'' = 0$ so there are no inflection points. Since the function is undefined at $x = 0$ and $x = 2$ there are still these three intervals to condsder for concavity: $(-\infty, 0), (0, 2), (2, \infty)$

For $x < 0$, $x^3 < 0$ so $f'(x) = -2/x^3 > 0$ so the function is concave up on $(-\infty, 0)$

For $x > 0$, $x^3 > 0$ so $f'(x) = -2/x^3 < 0$ so the function is concave down on $(0, 2)$ and $(2, \infty)$



2. $y = \dfrac{x^2}{x^2+9}$

Horizontal asymptote: $y = 1$

Vertical asymptote: none

Critical Point: Solve $y' = 0$ for x.

$$y' = \frac{18x}{(9+x^2)^2}$$

$$y' = 0 \implies x = 0$$

This gives two intervals, $(-\infty, 0)$ and $(0, \infty)$. Since the denominator is always positive, the sign of the slope is the same as the sign of x. Hence we have:

$(-\infty, 0)$: decreasing since $f' < 0$ everywhere on this interval.

$(0, \infty)$: increasing since $f' > 0$ everywhere on this interval.

To examine the concavity and the critical point, we take the second derivative.

$$y'' = \frac{54(3-x^2)}{(9+x^2)^3}$$

At the critical point $x = 0$ we have

$$y''(0) = \frac{2}{9} > 0$$

hence the function is concave up (and has a local minimum) at $x = 0$.

To find the inflection points we solve $y'' = 0$ for x.

$$0 = \frac{54(3-x^2)}{(9+x^2)^3} \implies x^2 = 3$$

Hence candidate inflection points are $\pm\sqrt{3}$.

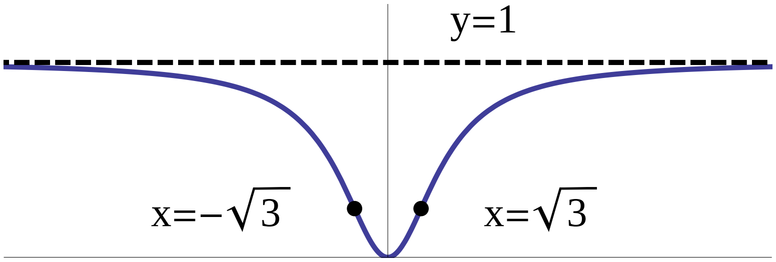
Looking at y'', we see that for large x the $y'' < 0$. This is because the denominator is positive for large x and $3 - x^2 < 0$ for large x. Hence $y'' = (-)/(+) < 0$. This means the function is concave down for large x. Since the concavity changes from up to down as we move from $x = 0$ to large x, the function must have an inflection point at $\sqrt{3}$.

A similar argument shows that $-\sqrt{3}$ is also an inflection point. We find that:

$(-\infty, -\sqrt{3})$: Concave Down

$(-\sqrt{3}, \sqrt{3})$: Concave Up

$(\sqrt{3}, \infty)$: Concave Down



3. $y = x^3 + 6x^2 + 9x$

This is a polynomial so there are no asymptotes.

Differentiating,

$$y' = 3x^2 + 12x + 9$$

$$y'' = 6x + 12$$

Critical points occur when

$$0 = y' = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3)$$

$$\implies 0 = (x+1)(x+3) \implies x = -1, -3$$

Since $y''(-1) = 6 > 0 \implies$ we conclude y is concave up (minimum) at $x = -1$

$y''(-3) = -6 < 0 \implies$ we conclude y is concave down (maximum) at $x = -3$

Thus the function satisfies:

$(-\infty, -3)$: increasing
$(-3, -1)$: decreasing
$(-1, \infty)$: increasing

Candidate inflection points occur when $y'' = 0$:

$$y'' = 0 \implies 6x + 12 = 0 \implies x = -2$$

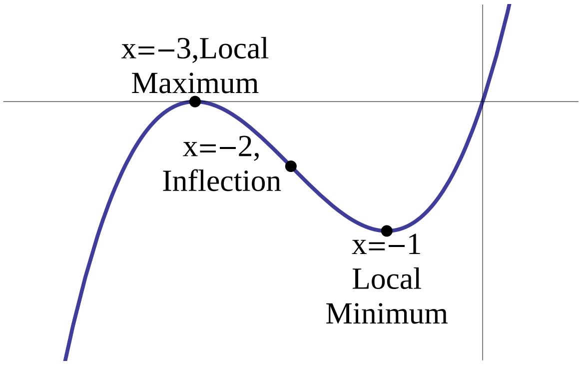
This divides the line into two intervals: $(-\infty, -2)$ and $(-2, \infty)$.

Since y is concave down at $x = -3$ and concave up at $x = -1$ and $x = -2$, the only candidate inflection point, lies between them, we conclude that there is an inflection point at $x = 2$.

The intervals of concavity are thus:

$(-\infty, -2)$: concave up

$(-2, \infty)$: concave down



4. $y = \dfrac{2x+4}{3x-1}$

Vertical Asymptote: $x = \dfrac{1}{3}$

Horizontal Asymptote: $y = \dfrac{2}{3}$

$x-$intercept: $x = -2$ (point (-2,0))

$y-$intercept: $y = -4$ (point (0, -4))