

Math 150A – Exam 4 Answers

1. Find the value of x for which the slope of the curve $y = 1 + 40x^3 - 3x^5$ is the largest.

We first need to find a formula for the slope. The slope of y is given by the derivative of y . Let $g(x)$ denote this slope. Then

$$g(x) = y' = 120x^2 - 15x^4$$

The function $g(x)$ is what we need to maximize. (NOT the function y).

To maximize $g(x)$ we calculate its first and second derivatives. (Note that these are not the first and second derivatives of y ; in fact, they are the second and third derivatives of y):

$$g'(x) = 240x - 60x^3$$

$$g''(x) = 240 - 180x^2$$

Critical points occur when $g'(x) = 0$:

$$0 = 240x - 60x^3 \implies 240x = 60x^3 \implies x = 0 \text{ or } x^2 = \frac{240}{60} = 4$$

So we have three critical points: $x = 0, x = 2, x = -2$.

$$g''(0) = 240 > 0 \implies \text{concave up, local minimum}$$

$$g''(\pm 2) = 240 - 180(2)^2 < 0 \implies \text{concave down, local maximum}$$

So the maximum slope occurs at one of the two local maximums of $g(x)$. Since $g(x)$ takes on the same value, maximums occur at both points:

$$g(2) = 120(2)^2 - 15(2)^4 = (120)(4) - 15(16) = 480 - 240 = 240$$

$$g(-2) = 120(-2)^2 - 15(-2)^4 = (120)(4) - 15(16) = 480 - 240 = 240$$

The maximum slope of $f(x)$ therefore occurs at both $x = 2$ and $x = -2$.

2. Find the area under the curve of $y = x(2 + \sqrt{x})$ and above the x -axis, on the interval $[1, 3]$. There will be a three point deduction if the answer is not exact (i.e., if it is a decimal approximation).

$$\begin{aligned} A &= \int_1^3 x(2 + \sqrt{x}) dx \\ &= \int_1^3 (2x + x^{3/2}) dx \\ &= 2\frac{x^2}{2} + \frac{x^{5/2}}{5/2} \Big|_1^3 \\ &= x^2 + \frac{2}{5}x^{5/2} \Big|_1^3 \\ &= 3^2 + \frac{2}{5}3^{5/2} - 1^2 - \frac{2}{5}1^{5/2} \\ &= 9 + \frac{2\sqrt{243}}{5} - 1 - \frac{2}{5} \\ &= \frac{(5)(8) - 2 + 2\sqrt{243}}{5} \\ &= \frac{38 + 2\sqrt{9 \times 9 \times 3}}{5} \\ &= \frac{2}{5} (19 + 9\sqrt{3}) \\ &\approx 1.38354 \end{aligned}$$

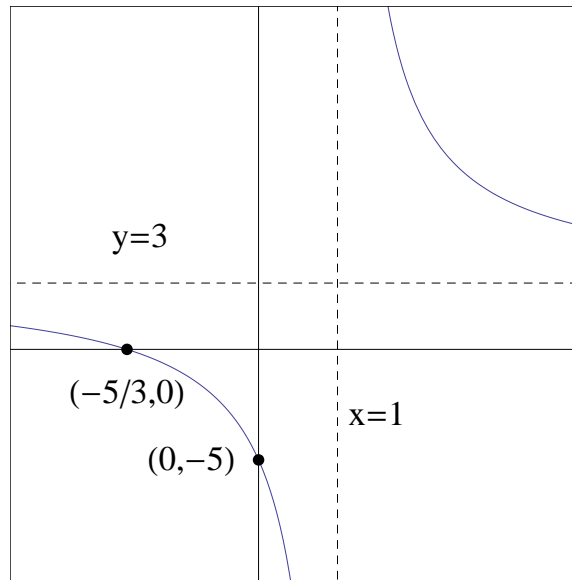
3. Sketch the curve $y = \frac{3x+5}{x-1}$, indicating the locations and values of all asymptotes and intercepts, if they exist.

horizontal asymptotes: $y = 3$ (take $\lim_{x \rightarrow \infty} y(x)$)

vertical asymptotes: $x = 1$ (denominator = 0)

y -intercepts: set $x = 0$: $y = -5$: intercept at $(0, -5)$

x -intercepts: set $y = 0$: $3x + 5 = 0 \implies x = -5/3$: intercept at $(-5/3, 0)$



4. Find the function $f(x)$ that satisfies $f''(x) = 1 - 6x + 48x^2$, $f'(1) = 16$, $f(1) = 0.5$.

$$y'' = 1 - 6x + 48x^2$$

$$y' = \int (1 - 6x + 48x^2) dx = x - 3x^2 + 16x^3 + C$$

$$16 = 1 - 3(1^2) + 16(1^3) + C = 14 + C \implies C = 2$$

$$y' = x - 3x^2 + 16x^3 + 2$$

$$y = \int (x - 3x^2 + 16x^3 + 2) dx = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + C$$

$$0.5 = \frac{1}{2}(1) - 1^3 + 4(1)^4 + 2(1) + C = 5.5 + C \implies C = -5$$

$$y = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x - 5$$

5. Let $f(x) = x - 2/x$.

(a) Write the Newton's method iteration formula for this problem.

(b) Use Newton's method with $x_0 = 2$ to find x_1 and x_2 . There will be a 3-point deduction if your answer is a decimal (calculator) approximation and not an exact fraction.

$$y = x - 2x^{-1} \implies y' = 1 + 2x^{-2} = 1 + 2/x^2$$

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} = x - \frac{x - 2/x}{1 + 2/x^2} \\ &= x - \frac{x^2(x - 2/x)}{x^2(1 + 2/x^2)} = x - \frac{x^3 - 2x}{x^2 + 2} \\ &= x \frac{x^2 + 2}{x^2 + 2} - \frac{x^3 - 2x}{x^2 + 2} = \frac{4x}{x^2 + 2} \end{aligned}$$

$$x_0 = 2$$

$$x_1 = \frac{4x_0}{x_0^2 + 2} = \frac{4(2)}{2^2 + 2} = \frac{8}{6} = \frac{4}{3} \approx 1.33$$

$$\begin{aligned} x_2 &= \frac{4x_1}{x_1^2 + 2} = \frac{4(4/3)}{(4/3)^2 + 2} = \frac{16/3}{16/9 + 2} \\ &= \frac{16(9)/3}{16(9)/9 + 2(9)} \\ &= \frac{48}{34} = \frac{24}{17} \approx 1.4117 \end{aligned}$$

6. A wire of length 10 is cut into two pieces. One is bent into a square with sides of length a , and other is bent into a circle of radius r . Find the value of r such that (a) the area enclosed is a minimum; and (b) the area enclosed is a maximum. You do not have to solve for the values of a or the areas. Three points will be deducted if the answers are not exact.

The area of the square is a^2 .

The area of the circle is πr^2 .

The total area is $A = a^2 + \pi r^2$

The perimeter of the square is $4a$.

The circumference of the circle is $2\pi r$.

The total amount of wire is $10 = 4a + 2\pi r \implies a = (10 - 2\pi r)/4 = (5 - \pi r)/2$

$$A = a^2 + \pi r^2 = \frac{(5 - \pi r)^2}{4} + \pi r^2$$

$$A' = \frac{2(5 - \pi r)(-\pi)}{4} + 2\pi r = -\frac{5\pi}{2} + \frac{\pi^2 r}{2} + 2\pi r$$

$$A'' = \frac{\pi^2}{2} + 2\pi > 0 \implies \text{concave up} \implies \text{crit. point. is minimum}$$

$$A' = 0 \implies \frac{5\pi}{2} = \frac{\pi^2 r}{2} + 2\pi r = \pi r \left(\frac{\pi}{2} + 2 \right)$$

$$r = \frac{5/2}{2 + \pi/2} = \frac{5}{4 + \pi} \approx 0.700124 \leftarrow \text{MINIMUM}$$

Since there is only one critical point, the maximum must occur at an endpoint. The endpoints are either (a) when all the wire is used for a square; or (b) when all the wire is used for a circle.

When all the wire is used for a square, then $r = 0$ and $a = 10/4 = 2.5$. Hence $A = a^2 = 2.5^2 = 6.25$.

When all the wire is used for the circle, then $r = 10/2\pi$ and $a = 0$. Then

$$A = \pi r^2 = \pi(10/2\pi)^2 = 25/\pi \approx 7.95775$$

Hence the MAXIMUM occurs when $r = 10/2\pi = 5/\pi \approx 1.59155$

7. Sketch the curve of $y = \frac{1}{3x^2 + 1}$, indicating all asymptotes (x and/or y values), local extrema (x values), and inflection points (both x and y values), if they exist.

Vertical asymptotes: none

Horizontal asymptotes: $y = 0$

Does the curve cross the asymptote: set $0 = y = \frac{1}{3x^2 + 1} \implies 0 = 1$ (no solution, so it does not cross).

$$y' = -\frac{6x}{(3x^2 + 1)^2}$$

$$y'' = -\frac{(3x^2 + 1)^2(6) - 6x(2)(3x^2 + 1)(6x)}{(3x^2 + 1)^4}$$

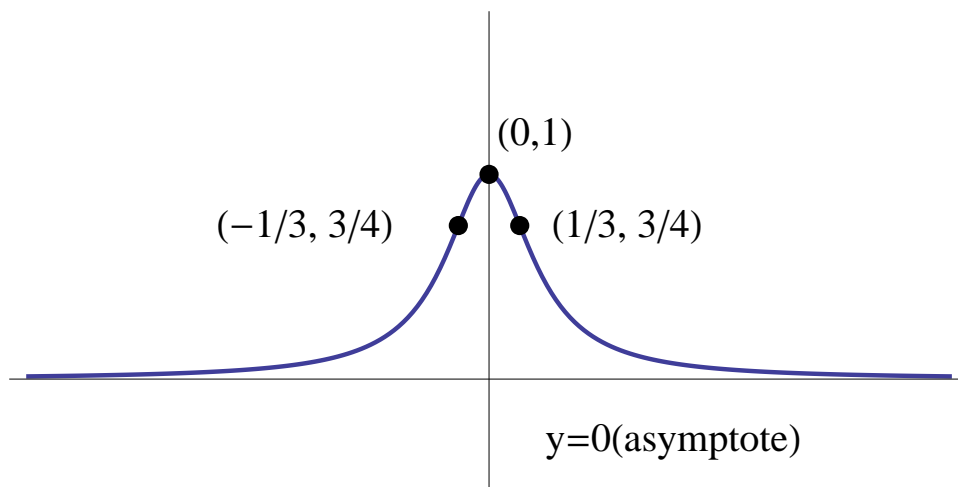
$$= -\frac{6(3x^2 + 1)}{(3x^2 + 1)^2}((3x^2 + 1) - 12x^2) = -\frac{6(1 - 9x^2)}{(3x^2 + 1)^3}$$

Critical point when $y' = 0 \implies x = 0$. $y''(0) < 0 \implies$ maximum at $x = 0$; $y(0) = 1$ so the maximum is at $(0, 1)$.

Inflection points occur when $y'' = 0 \implies 1 - 9x^2 = 0 \implies x = \pm 1/3$.

$$y(\pm 1/3) = \frac{1}{3(1/3)^2 + 1} = \frac{1}{1/3 + 1} = \frac{1}{4/3} = \frac{3}{4}$$

So the inflection points are at $(-1/3, 3/4)$ and $(1/3, 3/4)$.



8. Find the point (x, y) on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

The square of the distance from (x, y) to $(1, 4)$ is

$$\begin{aligned}f(x, y) &= (x - 1)^2 + (y - 4)^2 = (y^2/2 - 1)^2 + (y - 4)^2 \\f'(y) &= 2(y^2/2 - 1)(2y/2) + 2(y - 4) \\&= (y^2 - 2)y + 2y - 8 \\&= y^3 - 2y + 2y - 8 \\&= y^3 - 8 \\f''(y) &= 3y^2\end{aligned}$$

The minimum occurs when $f'(y) = 0$ and $f''(y) > 0$.

$$\begin{aligned}0 = f'(y) &\implies 0 = y^3 - 8 \implies y = 2 \\f''(2) &= 3(2)^2 > 0 \implies \text{minimum}\end{aligned}$$

So the minimum occurs when $y = 2$. Hence $x = y^2/2 = 2^2/2 = 2$. So the closest point on the parabola $y^2 = 2x$ to the point $(1, 4)$ is at $(2, 2)$.

9. Find the following integrals and simplify:

(a) $\int (x^3 - 2x + \cos x) dx$

$$\int (x^3 - 2x + \cos x) dx = \frac{1}{4}x^4 - x^2 + \sin x + C$$

(b) $\int \frac{x^4 + 3\sqrt{x}}{x^2} dx$

$$\begin{aligned} \int \frac{x^4 + 3\sqrt{x}}{x^2} dx &= \int (x^2 + 3x^{-3/2}) dx \\ &= \frac{x^3}{3} + 3 \frac{x^{-3/2+1}}{-3/2+1} + C \\ &= \frac{x^3}{3} + 3 \frac{x^{-1/2}}{-1/2} + C \\ &= \frac{x^3}{3} - \frac{6}{\sqrt{x}} + C \end{aligned}$$

10. Find the following integrals exactly:

(a) $\int_1^2 \frac{3}{t^4} dt$

$$\begin{aligned}\int_1^2 3t^{-4} &= \left. \frac{3t^{-4+1}}{-4+1} \right|_1^2 \\ &= \left. \frac{3t^{-3}}{-3} \right|_1^2 \\ &= \left. -\frac{1}{t^3} \right|_1^2 \\ &= -\frac{1}{2^3} - \left(-\frac{1}{1^3}\right) = \frac{7}{8}\end{aligned}$$

(b) $\int_{-\pi}^{\pi} (x^3 + \cos x) dx$

$$\begin{aligned}\int_{-\pi}^{\pi} (x^3 + \cos x) dx &= \left. \frac{x^4}{4} + \sin x \right|_{-\pi}^{\pi} \\ &= \frac{\pi^4}{4} + \sin \pi - \frac{(-\pi)^4}{4} - \sin(-\pi) \\ &= 0\end{aligned}$$